S.-T. Yau College Student Mathematics Contest
Applied and Computational Math (Team Contest)

June 9, 2024

Question 1

Let $\varphi : (0, \infty) \to (0, \infty)$ be continuous and increasing, and let $M > 0$. Given any T_* such that $0 < T_* < \int_M^\infty dz/\varphi(z)$, show that there exists $C_* > 0$ independent of $\Delta t > 0$ with the following property. Suppose that quantities $z_n, w_n \geq 0$ satisfy

$$z_n + \sum_{k=0}^{n-1} w_k \Delta t \leq y_n := M + \sum_{k=0}^{n-1} \varphi\left(z_k\right) \Delta t,$$

for $n = 0, 1, \ldots, n_*$, with $n_* \Delta t \leq T_*$. Then $y_{n_*} \leq C_*$ (independent of Δt).

Question 2

Let u be the solution to the transport equation

$$u_t + u_x = 0,$$

on the $0 \leq x \leq 2\pi$ with periodic boundary conditions and the initial data $u(x, 0) = \exp(-(x-\pi)^2)$. Consider the discrete approximation to the solution $v_j \approx u(x_j, t)$ on the grid $x_j = jh$, $h = \dfrac{2\pi}{N+1}$, $j = 1, \cdots, N+1$. We define three finite difference operators acting on a grid function v_j:

$$D_- v_j = \frac{v_j - v_{j-1}}{h}, \quad D_+ v_j = \frac{v_{j+1} - v_j}{h}, \quad D_0 v_j = \frac{v_{j+1} - v_{j-1}}{2h}\,.$$

(a) Consider the following three semi-discretizations

$$(i) \ \ \frac{dv_j}{dt} + D_- v_j = 0, \qquad (ii) \ \ \frac{dv_j}{dt} + D_+ v_j = 0, \qquad (iii) \ \ \frac{dv_j}{dt} + D_0 v_j = 0\,.$$

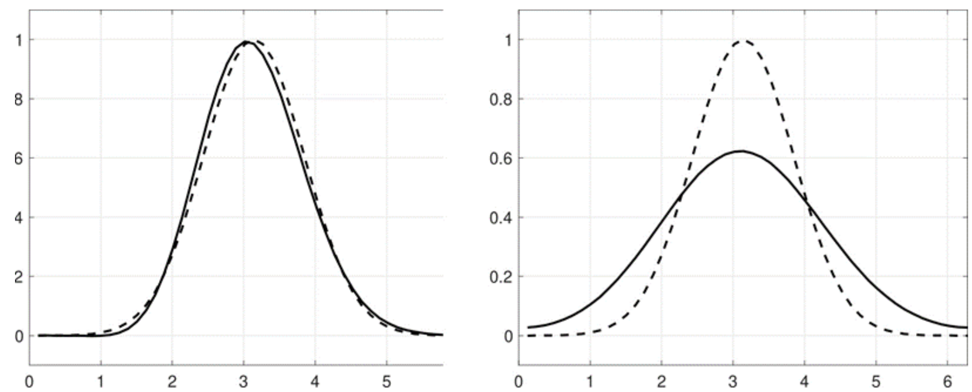
It is known that two of the above (i) - (iii) are stable and produce the results in Figure 1 when evolved one period in time using the classic fourth-order Runge-Kutta method.

(a-i). Which of the three semi-discretizations is not stable and why?

(a-ii). For the two stable methods, what method goes with which plot in Figure 1 and why?

(b) Let D denote one of the difference operators above. We take a forward Euler scheme in time to have

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + D\Big(\frac{v_j^{n+1} + v_j^n}{2}\Big) = 0$$



(A) (B)

Figure 1: Solid lines represent the numerical solutions, and dashed lines represent the exact solutions

where the superscript n now denotes the time index. Show that with this time-stepping, the spatial discretization corresponding to Figure 1 (A) satisfies

$$\|v^{n+1}\|_h^2 = \|v^n\|_h^2 \,,$$

while the discretization corresponding to Figure 1 (B) satisfies

$$\|v^{n+1}\|_h^2 \leq \|v^n\|_h^2 \,.$$

where the norm $\|v\|_h^2 = (v, v)_h$ with the inner product $(v, w)_h = \sum_{j=1}^{N+1} h v_j w_j$. *Hint: For the derivation of the last inequality, it might be useful to first find α_+ and/or α_- such that $D_\pm v_j = D_0 v_j + \alpha_\pm D_+ D_- v_j$.*

Question 3

Let u be a velocity field in $\mathbb{R}^3$ and let $\Gamma^u(t)$ be a surface which evolves under velocity filed u as time t increases, with a bounded closed initial surface $\Gamma^u(0) = \Gamma^0$. Namely, the surface is the image of a flow map $X^u(.,t) : \Gamma^0 \to \mathbb{R}^3$ satisfying the following differential equation:

$$\begin{aligned} \frac{\partial}{\partial t} X^u(p,t) &= u(X^u(p,t)) \quad \text{for } p \in \Gamma^0 \\ X^u(p,0) &= p \qquad\qquad\quad\; \text{for } p \in \Gamma^0. \end{aligned} \tag{1}$$

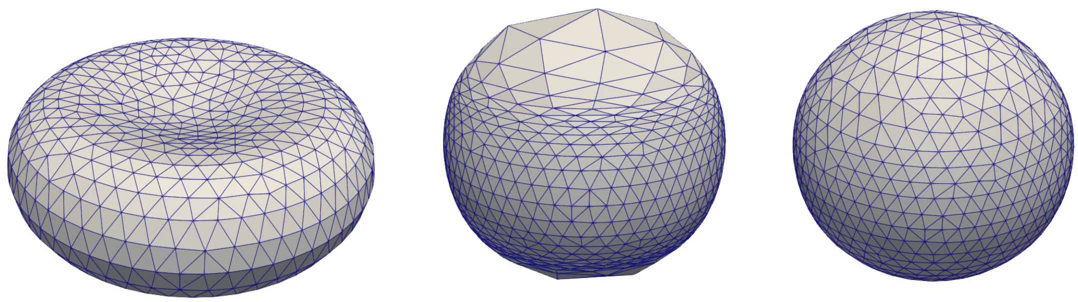
If one solves this ordinary differential equation (ODE) for a set of points p on the initial surface Γ^0, then one gets the location of these points at any time t. However, since the surface may have large deformation, the points obtained in this way may not form a good mesh for $\Gamma^u(t)$, as shown in Figure 2 (b).

David Gu and Shing-Tung Yau considered the following problem of finding a "good" map between two surfaces Γ^0 and $\Gamma^u(t)$ to produce good mesh on $\Gamma^u(t)$:

$$\text{Find } X(\cdot,t) : \Gamma^0 \to \Gamma^u(t) \text{ to minimize } \int_{\Gamma^0} |\nabla_{\Gamma^0} X|^2 \text{Ṣ}, \tag{2}$$

where ∇_{Γ^0} denotes surface tangential gradient and Ṣ denotes the surface area element. This map $X(\cdot,t) : \Gamma^0 \to \Gamma^u(t)$ minimizes the deformation energy $\int_{\Gamma^0} |\nabla_{\Gamma^0} X|^2$Ṣ and therefore could produce a better mesh (with less deformation). An efficient method of computing a good flow map $X(\cdot,t) : \Gamma^0 \to \mathbb{R}^3$ satisfying (2) is to solve the following problem:

$$\begin{aligned} v \cdot n &= u \cdot n \qquad\quad \text{on } \Gamma(t) \\ -\Delta_{\Gamma^0} X &= \kappa\,(n \circ X) \quad \text{on } \Gamma^0, \end{aligned} \tag{3}$$



(a) Mesh on initial surface Γ^0

(b) Surface at $t = 4$ by solving ODE (1)

(c) Surface at $t = 23$ by solving PDE (3)

Figure 2: Surface $\Gamma(t)$ computed by different numerical methods

where n denotes the unit normal vector on surface $\Gamma(t)$, where κ is an unknown scalar function to be solved. The method in (3) is based on the following ideas: The shape of an evolving surface is only determined by its normal velocity, rather than its tangential velocity. However, solving PDE (3) significantly improves the mesh quality, as shown in Figure 2 (c).

Problem (3) can be solved by a finite element method as follows. Let $t_m = m\tau$, $m = 0, 1, 2, \ldots$, with time stepsize τ, and suppose that Γ_h^m is a given triangulated surface which well approximates the bounded smooth closed surface $\Gamma(t_m)$. Let $X_h^m : \Gamma_h^0 \to \Gamma_h^m$ be the piecewise linear map which maps each triangle of Γ_h^0 to a triangle of Γ_h^m (thus X_h^m maps Γ_h^0 onto Γ_h^m), and let $N_h^m = n_h^m \circ X_h^m$ with n_h^m being the unit normal vector on Γ_h^m (piecewise constant vector defined on each triangle). Let $S_h^1(\Gamma_h^0)^3$ be the finite element space of 3-dimensional vector-valued piecewise linear functions on Γ_h^0, and let $S_h^0(\Gamma_h^0)$ be the finite element space of scalar-valued piecewise constant functions on Γ_h^0. Find $(X_h^{m+1}, \kappa_h^{m+1}) \in S_h^1(\Gamma_h^0)^3 \times S_h^0(\Gamma_h^0)$ such that

$$\begin{aligned}
\int_{\Gamma_h^0} \frac{X_h^{m+1} - X_h^m}{\tau} \cdot N_h^m \chi_h \text{Ṣ} &= \int_{\Gamma_h^0} (u \circ X_h^m) \cdot N_h^m \chi_h \text{Ṣ} \quad \forall \chi_h \in S_h(\Gamma_h^0) \\
\int_{\Gamma_h^0} \nabla_{\Gamma_h^0} X_h^{m+1} \cdot \nabla_{\Gamma_h^0} v_h \text{Ṣ} &= \int_{\Gamma_h^0} \kappa_h^{m+1} N_h^m \cdot v_h \text{Ṣ} \qquad \forall v_h \in S_h(\Gamma_h^0)^3.
\end{aligned} \tag{4}$$

(i) Show that the map X determined by (3) is a local minimizer of (2) with $\Gamma^u(t)$ determined by (1).

(ii) Show that the weak formulation in (4) has a unique solution $(X_h^{m+1}, \kappa_h^{m+1}) \in S_h^1(\Gamma_h^0)^3 \times S_h^0(\Gamma_h^0)$.